Numerical Computation of the Effective Potential and Renormalization

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Abstract

We present a novel way to compute the one-loop ring-improved effective potential numerically, which avoids the spurious appearence of complex expressions and at the same time is free from the renormalization ambiguities of the self-consistent approaches, based on the direct application of Schwinger-Dyson type equations to the masses.

1 Introduction

In the study of the nature of the electroweak phase transition in the early Universe the effective potential plays an essential role. Since the early studies it was realized that the computation of the effective potential was plagued with difficulties which are mainly related with the bad infrared behavior of the theory near the critical temperature [1]. In order to avoid the appearance of imaginary terms in the effective potential it was then necessary to improve its perturbative computation by resumming an infinite subset of the leading infrared divergent diagrams, called the ring diagrams. An important improvement was made by the use of self-consistent methods based on Schwinger-Dyson equations for the masses. They can be obtained through a modification of the mass term in the original Lagrangian; a mass parameter is introduced in the beginning and later determined by consistency conditions, the gap equations (see e.g. [2],[3]).

In spite of the considerable progress made in this direction, there still remain some ambiguities in the calculation of the effective potential at finite temperature which are mainly related with the correct way to perform the resummation of the infrared divergent diagrams. Also when using the kind of methods based on gap equations it will always appear temperature dependent ultraviolet divergencies, which implies that in order to renormalize the theory one

should include temperature dependent counterterms. Although these counterterms have the same structure as those of the $T = 0$ theory, its physical origin is very unclear.

In this letter we show a novel way of computing the one-loop ring-improved effective potential numerically which avoid the above mentioned problems related with renormalization. It is enough to consider the $\lambda\phi^4$ -theory to explain the method, but for definitness the final calculation is carried out within the Standard Model.

2 The self-consistent way

We begin by reviewing (in the $\lambda\phi^4$-theory) the usual way one performs the improvement of the one-loop effective potential at finite temperature. We add and substract in the original lagrangian the induced thermal mass $M^2 = \Pi_T(\omega_n, p)$, where $\omega_n = 2\pi nT$, $n = 0, \pm 1, ...$ are the Matsubara frequencies. The result of this procedure is an improved perturbation theory in which the original propagator is replaced by the full propagator. The polarization tensor $\Pi_T(\omega_n, p)$ can be obtained consistently within perturbation theory. Due to the fact that it is the infrared behavior the relevant aspect near the phase transition, it is only necessary to compute Π_T in the infrared limit $\omega_n = 0$, $p \to 0$, $\Pi_T(0)$. This procedure amounts to a summation of an infinite set of diagrams with the worst infrared behavior, called the ring diagrams, and which are generated by including polarization tensor insertions in the infrared limit in one loop diagrams. It is not necessary to do that for fermions since in this case the Matsubara frequencies are always different from zero. Therefore, the effective potential reads

$$U_{eff,T}(\phi) = U_{cl}(\phi) + \frac{T}{2}\sum_n \int \frac{d^3k}{(2\pi)^3} \ln\left[\omega_n^2 + \vec{\mathbf{k}}^2 + m_{pl}^2(\phi)\right], \tag{1}$$

where $m_{pl}^2(\phi) = m^2(\phi) + \Pi_T(0)$ is the plasma mass ($m^2(\phi) = -\nu^2 + \lambda\phi^2/2$). The sum over Matsubara frequencies can be performed and one obtains

$$U_{eff,T}(\phi) = U_{cl}(\phi) + \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\sqrt{\vec{\mathbf{k}}^2 + m_{pl}^2(\phi)} + \frac{T}{2\pi^2}\int d k \vec{\mathbf{k}}^2 \ln\left[1 - e^{-\sqrt{\vec{\mathbf{k}}^2 + m_{pl}^2(\phi)}/T}\right] \tag{2}$$

It is clear that the second term on the r.hs. of this expression contains temperature dependent ultraviolet divergent contributions. Therefore temperature dependent counterterms are needed in order to renormalize this expression. In spite of the similar structure of these counterterms with those of the $T = 0$ limit, its physical origin is not clear and represents a lack of consistency of the procedure used [4]. In fact the renormalization problem is related with the local behavior

of the theory and should not be influenced by finite temperature effects, which concerns long distance physics.

Nevertheless if one accepts to work with temperature dependent counterterms one can formally derive an expression for the effective potential. To find explicit expressions see ref. [5].

3 Our proposal

In this section we show a novel way of computing the one-loop ring-improved effective potential which avoids the above mentioned problems related with renormalization.

In this approach we will consider the improvement of the effective potential by the summation of rings diagrams, which in this case are generated by including polarization tensor insertions in the infrared limit, in one-loop diagrams with *zero Matsubara frequencies*. To explain the method we start with the expression for the one-loop ring-improved effective potential

$$\hat{U}_{eff,T}(\phi) = -\frac{1}{2}\nu^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{T}{2}\int \frac{d^3k}{(2\pi)^3} \ln\left[\vec{\mathbf{k}}^2 + m_{pl}^2(\phi)\right], \tag{3}$$

where the third term on the r.h.s can be separated into the one-loop

$$\hat{U}^{(1)}_{eff,T}(\phi) = \frac{T}{2}\int \frac{d^3k}{(2\pi)^3} \ln\left[\vec{\mathbf{k}}^2 + m^2(\phi)\right] \tag{4}$$

and the ring

$$\hat{U}^{ring}_{eff,T}(\phi) = -\frac{T}{12\pi}\left(\left[m^2(\phi) + \Pi_T(0)\right]^{3/2} - m^3(\phi)\right) \tag{5}$$

contributions.

At first sight, by looking at eqs. (4) and (5) one should be tempted to say that there is a difficulty in using the above equation for the effective potential, because for small enough values of the classical field ϕ the scalar mass squared $m^2(\phi)$ is negative and therefore the potential itself would be a complex expression in that range. Of course this is not so because although both $\hat{U}^{(1)}_{eff,T}(\phi)$ and $\hat{U}^{ring}_{eff,T}(\phi)$ become complex for small enough values of ϕ, their sum, given by equation (3), is real provided $m^2_{pl}(\phi) > 0$. This is not new and people have used this fact to evaluate the effective potential in the high-temperature expansion. In this case the "dangerous" terms, i.e. those which become imaginary if $m^2(\phi) < 0$ cancel each other and everything is safe.

But what happens if we want to evaluate the one-loop ring-improved effective potential in an exact way, that is, numerically? In this case a calculational problem arises because the

scalar mass squared are negative for finite values of ϕ. The usual way out of this problem is to carry out the self consistent method outlined in the previous section which, as we have seen, from first principles is not correct, because it mixes the infrared and ultraviolet regimes.

The method we propose is based on eq. (3) and avoids all the above mentioned difficulties. The important point to remember is that $\hat{U}_{eff,T}(\phi)$ is real provided $m^2_{pl}(\phi) > 0$.

The one-loop contribution to the effective potential $\hat{U}_{eff,T}(\phi)$ is

$$\begin{aligned}\hat{U}^{(1)}_{eff,T}(\phi) &= \frac{1}{64\pi^2}\left\{m^4(\phi)\left(\ln\left[\frac{m^2(\phi)}{m^2(\phi_0)}\right]-\frac{3}{2}\right)+2m^2(\phi)m^2(\phi_0)\right\} \\ &+\frac{T^4}{2\pi^2}\int_0^\infty dx x^2 \ln\left[1-e^{-\sqrt{x^2+m^2(\phi)/T^2}}\right]. \end{aligned} \tag{6}$$

The renormalization is performed here at zero temperature which in turns implies that the required counterterms are temperature-independent. As renormalization prescription we require that both the classical values of the Higgs boson mass $m^2 = 2\nu^2$ and the Higgs field at the minimum of the effective potential at $T = 0$, $\phi_0 = \sqrt{6\nu^2/\lambda}$, to be preserved even after quantum corrections are taken into account. In opposition to what happens in the self-consistent approach, the renormalization procedure is free of any theoretical inconsistency.

If $m^2(\phi) > 0$ everything is standard and no difficulties appear in the calculation of eq. (6) . Therefore let us consider the case when $m^2(\phi) < 0$ and define $\widehat{m}^2(\phi) \equiv -m^2(\phi) > 0$. Thus, by separating the range of integration in the temperature dependent part we have

$$\begin{aligned}\hat{U}^{(1)}_{eff,T}(\phi) &= \frac{T^4}{2\pi^2}\int_0^{\widehat{m}(\phi)/T} dx x^2 \ln\left[1-e^{-i\sqrt{\widehat{m}^2(\phi)/T^2-x^2}}\right] \\ &+\frac{T^4}{2\pi^2}\int_{\widehat{m}(\phi)/T}^\infty dx x^2 \ln\left[1-e^{-\sqrt{x^2-\widehat{m}^2(\phi)/T^2}}\right]. \end{aligned} \tag{7}$$

Clearly the second term is real, while the first one is complex and can be explicited as follows. Notice that

$$\begin{aligned}\ln\left[1-e^{-i\sqrt{\widehat{m}^2(\phi)/T^2-x^2}}\right] &= -i\sqrt{\widehat{m}^2(\phi)/T^2-x^2}/2+\ln 2+i\frac{\pi}{2} \\ &+\ln\left[\sin\left(\frac{1}{2}\sqrt{\widehat{m}^2(\phi)/T^2-x^2}\right)\right], \end{aligned} \tag{8}$$

where we have chosen the branch point where $\ln i = i\pi/2$. Thus,

$$\begin{aligned}\hat{U}^{(1)}_{eff,T}(\phi) &= -i\frac{T^4}{2\pi^2}\int_0^{\widehat{m}(\phi)/T} dx x^2\left[\sqrt{\widehat{m}^2(\phi)/T^2 - x^2} - \pi\right] \\ &+\frac{T^4}{2\pi^2}\int_0^{\widehat{m}(\phi)/T} dx x^2 \ln\left[2\sin\left(\frac{1}{2}\sqrt{\widehat{m}^2(\phi)/T^2 - x^2}\right)\right] \\ &+\frac{T^4}{2\pi^2}\int_{\widehat{m}(\phi)/T}^{\infty} dx x^2 \ln\left[1 - e^{-\sqrt{x^2 - \widehat{m}^2(\phi)/T^2}}\right]. \end{aligned} \tag{9}$$

The imaginary part of $\hat{U}^{(1)}_{eff,T}(\phi)$ can be integrated exactly and the result is

$$Im\hat{U}^{(1)}_{eff,T}(\phi) = -\frac{1}{64\pi^2}\widehat{m}^4(\phi) + \frac{T}{12\pi}\widehat{m}^3(\phi)$$

Notice that the second term on the r.h.s. corresponds to one of the terms appearing in the ring contribution in the scalar sector, see eq. (5). Also notice that the presence of the extra term, which does not appear in the ring contribution. Its effect is to cancel the imaginary part of the zero temperature contribution coming from the logarithm (remember that we are considering the case of negative mass squared).

This shows an exact cancellation of the spurious imaginary terms in the effective potential, as it should be the case because for $m^2_{pl}(\phi) > 0$, the effective potential is real . Thus, it is clear that such a cancellation does not depend on the order of the high-temperature expansion.

Therefore, the final expression for the effective potential, for the case $m^2(\phi) < 0$, is

$$\begin{aligned}\hat{U}_{eff,T}(\phi) &= -\frac{1}{2}\nu^2\phi^2 + \frac{\lambda}{4!}\phi^4 + \frac{1}{64\pi^2}\left\{\widehat{m}^4(\phi)\left(\ln\left[\frac{\widehat{m}^2(\phi)}{m^2(\phi_0)}\right] - \frac{3}{2}\right) - 2\widehat{m}^2(\phi)m^2(\phi_0)\right\} \\ &+\frac{T^4}{2\pi^2}\int_0^{\widehat{m}(\phi)/T} dx x^2 \ln\left[2\sin\left(\frac{1}{2}\sqrt{\widehat{m}^2(\phi)/T^2 - x^2}\right)\right] \\ &+\frac{T^4}{2\pi^2}\int_{\widehat{m}(\phi)/T}^{\infty} dx x^2 \ln\left[1 - e^{-\sqrt{x^2 - \widehat{m}^2(\phi)/T^2}}\right]. \end{aligned} \tag{10}$$

4 Numerical results

As we said in the introduction, we present our numerical results in the context of the Standard Model. The values of the SU(2) and U(1) couplings constants are $g_2 = 0.647$ and $g_1 = 0.344$, respectively. Also, we take $\phi_0 = 246$ GeV.

We implement the first (self-consistent) approach by following Carrington [5], that is, by including the whole range of Matsubara frequencies in the scalar sector and by considering only the contribution of the zero Matsubara frequency in the gauge sector. In particular, this means that we consider equations of the form (2) in order to evaluate the temperature dependent part of the effective potential. More precisely, we use the same set of equations that she used in order to evaluate the effective potential[1].

The way we perform the numerical computation of the effective potential using our approach is to separate the cases where the squared mass of the Higgs boson (respectively the Goldstone boson) is positive or zero from that where it is negative. In the first case, no problems appear in the numerical evaluation of the effective potential, while we use expressions like equation (10) when the scalar masses squared become negative. The rest of the expressions we use for the evaluation of the effective potential are those given in ref. [5].

Figure 1 shows the effective potential at the critical temperature, for a top-quark of mass 180 GeV, versus the value of ϕ. The appearence of a first order phase transition is shown. Notice that both results coincide considerable well in a wide range, the differences appearing for values of ϕ bigger than the position of the non trivial minimum of the effective potential. Indeed the critical temperature, defined when the minima of the effective potential are degenerate, is the same when calculated through the two different ways. Moreover, the ratio ϕ/T at the critical temperature, which is the relevant parameter to be studied in order to conclude something about the baryon asymmetry washout, is unaltered in both calculations as Figure 2 shows.

Since both methods of evaluation of the effective potential differ by the way how the Matsubara frequencies are taken into account (considering all Matsubara frequencies in the first method and only the zero mode in our proposal), the results show in a numerical way the irrelevance of those terms contributed by the nonzero Matsubara frequencies.

Although our results do not differ significatively from those obtained through the application of the self-consistent approach, our method has the advantage of making the renormalization procedure conceptually clear, avoiding the necessity of introducing temperature dependent counterterms.

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[1]Our numerical results do not coincide with those of Carrigton when using the same equations. Since we have checked the correctnes of our calculations, we conclude that the difference must be related to a mistake in her numerical computation.

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Figure Captions

Figure 1. The one-loop ring improved effective potential at finite temperature U(ϕ) versus ϕ is shown, for a Higgs mass of 60 GeV and a top-quark mass of 180 GeV. Circles represent the data from the self-consistent method, while crosses are data from our proposal.

Figure 2. The ratio ϕ/T for the same parameters as in figure 1 is plotted as a function of the Higgs mass. Notation is as in figure 1.

Fig. 1

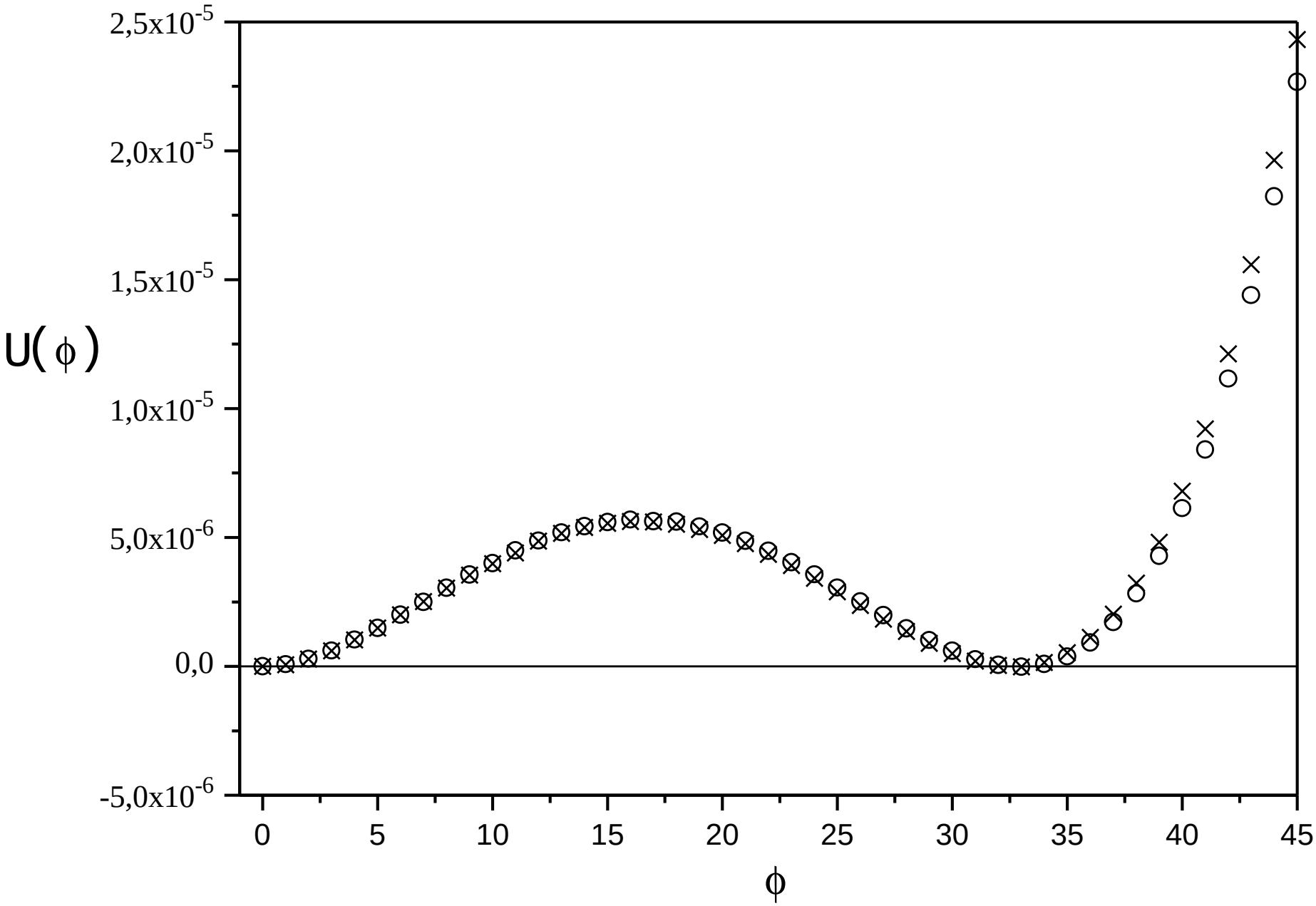
2,5x10-5
2,0x10-5
1,5x10-5
1,0x10-5
5,0x10-6
0,0
-5,0x10-6
U(ϕ)
0
5
10
15
20
25
30
35
40
45
ϕ

Fig. 2